

Two-spinor geometry and gauge freedom

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Abstract

Gauge freedom in quantum particle physics is shown to arise in a natural way from the geometry of two-spinors (Weyl spinors). Various related mathematical notions are reviewed, and a special ansatz of the kind “the system defines the geometry” is discussed in connection with the stated results.

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Introduction

ENTIA NON SVNT MVLTIPPLICANDA PRAETER NECESSITATEM

(Entities are not to be multiplied beyond necessity)

William of Ockam

In previous papers I discussed some partly original notions related to quantum particle physics, including a “minimal geometric data”¹ approach to in 2-spinor geometry and field theories [3, 4, 5, 6, 9, 11], the geometry of distributional bundles in terms of Frölicher smoothness [7] and its application to quantum bundles, quantum connections and particle interactions [8, 10, 13]. While these mathematical results have been offered as they are, and perhaps can be read as small bits of clarification about a rather confused matter, they are actually driven by a somewhat radical ansatz, related in part to ideas once proposed by Penrose [22], which was exposed in an essay [12] presented for the 2011 contest of the Foundational Questions Institute.

Section 1 is devoted to a sketch of 2-spinor geometry and the treatment of gauge field theories based on it, together with a somewhat novel discussion of symmetry breaking. The notions of quantum bundles, quantum states and quantum interactions are reviewed in section 2. The paper’s main section is §3, where the notion of gauge freedom is looked at from different points of view, and, in particular, in terms of 2-spinor geometry. Finally, some foundational issues related to the exposed ideas are discussed; there I do not claim to have demonstrated my main thesis, but argue that some clues support it.

1 Two-spinors and gauge field theory

1.1 Two-spinor basics

If V is a complex vector space then Hermitian transposition is a natural anti-linear involution of $V \otimes \overline{V}$ (where \overline{V} denotes the conjugate space), determining a decomposition into the direct sum of the *real* eigenspaces corresponding to eigenvalues ± 1 , namely

$$V \otimes \overline{V} = H(V \otimes \overline{V}) \oplus i H(V \otimes \overline{V}) ,$$

called the *Hermitian* and *anti-Hermitian* subspaces, respectively

Starting from a 2-dimensional complex vector space S , with no further assumption, the above basic construction gives rise to a rich algebraic structure:

- The Hermitian subspace of $\wedge^2 S \otimes \wedge^2 \overline{S}$ is a real 1-dimensional vector space with a distinguished orientation; its positively oriented semispace \mathbb{L}^2 (whose elements are of the type $w \otimes \bar{w}$, $w \in \wedge^2 S$) has the square root semispace \mathbb{L} , which will can be identified with the space of *length units* [17, 13].
- The 2-spinor space is defined to be $U := \mathbb{L}^{-1/2} \otimes S$. The space $\wedge^2 U$ is naturally endowed with a Hermitian metric, namely the identity element in

$$H[(\wedge^2 \overline{U}^\star) \otimes (\wedge^2 U^\star)] \cong \mathbb{L}^2 \otimes H[(\wedge^2 \overline{S}^\star) \otimes (\wedge^2 S^\star)] ,$$

so that normalised ‘symplectic forms’ $\varepsilon \in \wedge^2 U^\star$ constitute a $U(1)$ -space (any two of them are related by a phase factor). Each ε yields the isomorphism $\varepsilon^\flat : U \rightarrow U^\star : u \mapsto u^\flat := \varepsilon(u, _)$.

¹This locution was crafted by A. Jadczyk, with whom I collaborated in these topics.

• The identity element in $H[(\wedge^2 \bar{\mathbf{U}}^\star) \otimes (\wedge^2 \mathbf{U}^\star)]$ can be written as $\varepsilon \otimes \bar{\varepsilon}$ where $\varepsilon \in \wedge^2 \mathbf{U}^\star$ is any normalised element. This natural object can also be seen as a bilinear form g on $\mathbf{U} \otimes \bar{\mathbf{U}}$, via the rule $g(u \otimes \bar{v}, r \otimes \bar{s}) = \varepsilon(u, r) \bar{\varepsilon}(\bar{v}, \bar{s})$ extended by linearity. Its restriction to the Hermitian subspace $\mathbf{H} \equiv H(\mathbf{U} \otimes \bar{\mathbf{U}})$ turns out to be a Lorentz metric. Null elements in \mathbf{H} are of the form $\pm u \otimes \bar{u}$ with $u \in \mathbf{U}$ (thus there is a distinguished time-orientation in \mathbf{H}).

• Let $\mathbf{W} \equiv \mathbf{U} \oplus \bar{\mathbf{U}}^\star$. The linear map $\gamma : \mathbf{U} \otimes \bar{\mathbf{U}} \rightarrow \text{End}(\mathbf{W}) : y \mapsto \gamma[y]$ acting as

$$\tilde{\gamma}(r \otimes \bar{s})[u, \bar{\lambda}] = \sqrt{2}(\langle \bar{\lambda}, \bar{s} \rangle p, \langle r^b, u \rangle \bar{s}^b)$$

is well-defined independently of the choice of the normalised $\varepsilon \in \wedge^2 \mathbf{U}^\star$ yielding the isomorphism ε^b . Its restriction to \mathbf{H} turns out to be a Clifford map. Thus one is led to regard $\mathbf{W} \equiv \mathbf{U} \oplus \bar{\mathbf{U}}^\star$ as the space of Dirac spinors, decomposed into its Weyl subspaces. The anti-isomorphism $\mathbf{W} \rightarrow \mathbf{W}^\star : (u, \bar{\lambda}) \mapsto (\lambda, \bar{u})$ is the usual *Dirac adjunction* ($\psi \mapsto \bar{\psi}$ in traditional notation), associated with a Hermitian product having the signature $(+, +, -, -)$.

An arbitrary basis (ζ_A) of \mathbf{S} , $A=1,2$, determines bases of the various associated spaces, in particular the bases $l \in \mathbb{L}$ (a length unit), $(\zeta_A) \equiv (l^{-1/2} \zeta_A) \subset \mathbf{U}$, $\varepsilon \in \wedge^2 \mathbf{U}^\star$. We have $\varepsilon = \varepsilon_{AB} \zeta^A \wedge \zeta^B$, where $(\zeta^A) \subset \mathbf{U}^\star$ is the dual basis of (ζ_A) and (ε_{AB}) denotes the antisymmetric Ricci matrix. As for the basis of $\mathbf{H} \equiv H(\mathbf{U} \otimes \bar{\mathbf{U}})$ associated with (ζ_A) one usually considers the *Pauli basis* (τ_λ) , given by $\tau_\lambda \equiv \frac{1}{\sqrt{2}} \sigma_\lambda^{AA'} \zeta_A \otimes \bar{\zeta}_{A'}$ where $(\sigma_\lambda^{AA'})$, $\lambda = 0, 1, 2, 3$, denotes the λ -th Pauli matrix (dotted indices refer to components in conjugate spaces). This basis is readily seen to be g -orthonormal. The associated *Weyl basis* of \mathbf{W} is defined to be the basis (ζ_α) , $\alpha = 1, 2, 3, 4$, given by

$$(\zeta_1, \zeta_2, \zeta_3, \zeta_4) := (\zeta_1, \zeta_2, -\bar{\zeta}^1, -\bar{\zeta}^2),$$

where ζ_1 is a simplified notation for $(\zeta_1, 0)$, and the like.

Remark. In contrast to the usual 2-spinor formalism, no symplectic form is fixed. The 2-form ε is unique up to a phase factor which depends on the chosen 2-spinor basis, and determines isomorphisms $\varepsilon^b : \mathbf{U} \rightarrow \mathbf{U}^\star$ and $\varepsilon^\# : \mathbf{U}^\star \rightarrow \mathbf{U}$. Also note that no Hermitian form on \mathbf{S} or \mathbf{U} is assigned; actually, because of the Lorentz structure of \mathbf{H} , the choice of such an object turns out to be equivalent to the choice of an ‘observer’.

We now consider a complex vector bundle $\mathbf{S} \rightarrow \mathbf{M}$ with 2-dimensional fibers. By performing the above sketched constructions fiberwise we obtain various vector bundles, which are denoted, for simplicity, by the corresponding symbols. We observe that some appropriate topological restrictions are implicit in what follows; we’ll assume the needed hypotheses to hold without further comment.

A linear connection F on \mathbf{S} determines linear connections on the associated bundles, and, in particular, connections G of \mathbb{L} , Y of $\wedge^2 \mathbf{U}$ and $\tilde{\Gamma}$ of \mathbf{H} ; on turn, it can be expressed in terms of these as

$$F_{aB}^A = (G_a + i Y_a) \delta_B^A + \frac{1}{2} \tilde{\Gamma}_a^{AA'} \delta_{BA'}.$$

If \mathbf{M} is 4-dimensional, then a *tetrad* is defined to be a linear morphism $\Theta : T\mathbf{M} \rightarrow \mathbb{L} \otimes \mathbf{H}$. An invertible tetrad determines, by pull-back, a Lorentz metric on \mathbf{M} and a metric connection of $T\mathbf{M} \rightarrow \mathbf{M}$, as well as a Dirac morphism $T\mathbf{M} \rightarrow \mathbb{L} \otimes \text{End } \mathbf{W}$.

A non-singular field theory in the above geometric environment can be naturally formulated [5] even if Θ is not required to be invertible everywhere. If the invertibility requirement is satisfied then one gets essentially the standard Einstein-Cartan-Maxwell-Dirac theory, but with some redefinition of the fundamental fields: these are now the 2-spinor connection F , the

tetrad Θ , the Maxwell field F and the Dirac field $\psi : \mathbf{M} \rightarrow \mathbb{L}^{-3/2} \otimes \mathbf{W}$. Gravitation is represented by Θ and $\tilde{\Gamma}$ together. G is assumed to have vanishing curvature, $dG = 0$, so that we can find local charts such that $G_a = 0$; this amounts to ‘gauging away’ the conformal ‘dilaton’ symmetry. Coupling constants arise as covariantly constant sections of \mathbb{L}^r (r rational). One then writes a natural Lagrangian which yields all the field equations: the Einstein equation and the equation for torsion; the equation $F = 2 dY$ (thus Y is essentially the electromagnetic potential) and the other Maxwell equation; the Dirac equation [6].

On the other hand, by fixing the tetrad Θ and the gravitational part of the spin connection one works in a fixed curved background structure. This is the setting of this paper and my previous articles about quantum theory. Then Θ allows the identification $T\mathbf{M} \cong \mathbb{L} \otimes \mathbf{H}$, and 1-forms of \mathbf{M} can be viewed as *scaled* sections $\mathbf{M} \rightarrow \mathbb{L}^{-1} \otimes \mathbf{H}^*$.

1.2 Gauge theories

The two-spinor treatment of electrodynamics suggests a natural procedure of generating gauge theories in some generality, though not in *all* generality: we’ll be able to recover the standard model and some possible extensions. In particular, we do not aim at a theory in which gravitation is on the same footing as other fields (see §3.4). Also note that, in our approach, the role of spinors is quite special, not at all analogous to other internal degrees of freedom.

Our starting assumption is that fermion fields can be described as sections of a vector bundle $\mathbf{Y} \rightarrow \mathbf{M}$ where

$$\mathbf{Y} \equiv \mathbf{Y}_R \oplus \mathbf{Y}_L \equiv (\mathbf{F}_R \otimes \mathbf{U}) \oplus (\mathbf{F}_L \otimes \overline{\mathbf{U}}^\star),$$

and where $\mathbf{F}_R \rightarrow \mathbf{M}$ and $\mathbf{F}_L \rightarrow \mathbf{M}$ are complex vector bundles, describing the internal fermion structure besides spin, endowed with fibered Hermitian structures (fibered tensor products and direct sums over \mathbf{M}). Next, by expanding $\overline{\mathbf{Y}} \otimes \mathbf{Y}$, we’ll notice that its sectors are natural candidates for the role of boson bundles. Further possible sectors may arise from the expansion of other tensor products. Because of the algebraic structure of the fibers one gets various contractions among different sectors, which we view as related to the possible particle interactions. Roughly speaking, the various tensor factors could be seen as an analogue of “chemical bonds”.

Explicitly, the expansion of our candidate boson bundle yields

$$\begin{aligned} \overline{\mathbf{Y}} \otimes \mathbf{Y} &\cong (\overline{\mathbf{Y}}_R \otimes \mathbf{Y}_R) \oplus (\overline{\mathbf{Y}}_L \otimes \mathbf{Y}_L) \oplus (\overline{\mathbf{Y}}_R \otimes \mathbf{Y}_L) \oplus (\overline{\mathbf{Y}}_L \otimes \mathbf{Y}_R) \cong \\ &\cong (\overline{\mathbf{F}}_R \otimes \mathbf{F}_R \otimes \overline{\mathbf{U}} \otimes \mathbf{U}) \oplus (\overline{\mathbf{F}}_L \otimes \mathbf{F}_L \otimes \mathbf{U}^\star \otimes \overline{\mathbf{U}}^\star) \oplus \\ &\oplus (\overline{\mathbf{F}}_R \otimes \mathbf{F}_L \otimes \overline{\mathbf{U}} \otimes \overline{\mathbf{U}}^\star) \oplus (\overline{\mathbf{F}}_L \otimes \mathbf{F}_R \otimes \mathbf{U}^\star \otimes \mathbf{U}). \end{aligned}$$

The Hermitian structures of \mathbf{F}_R and \mathbf{F}_L yield fibered isomorphisms $\overline{\mathbf{F}}_R \cong \mathbf{F}_R^\star$ and $\overline{\mathbf{F}}_L \cong \mathbf{F}_L^\star$; we also have $\mathbf{U} \otimes \overline{\mathbf{U}} \cong \mathbb{C} \otimes \mathbf{H}$, $\mathbf{U}^\star \otimes \overline{\mathbf{U}}^\star \cong \mathbb{C} \otimes \mathbf{H}^*$; the Lorentz metric yields the isomorphism $\mathbf{H} \leftrightarrow \mathbf{H}^*$, and the tetrad Θ yields the scaled isomorphism $\mathbf{H}^* \leftrightarrow \mathbb{L} \otimes T^*\mathbf{M}$. Hence, after rearranging the order of tensor factors, sections $\mathbf{M} \rightarrow \mathbb{L}^{-1} \otimes \overline{\mathbf{Y}}_R \otimes \mathbf{Y}_R$ and $\mathbf{M} \rightarrow \mathbb{L}^{-1} \otimes \overline{\mathbf{Y}}_L \otimes \mathbf{Y}_L$ can be seen as fields $\mathbf{M} \rightarrow T^*\mathbf{M} \otimes \mathbf{F}_R \otimes \mathbf{F}_R^\star$ and $\mathbf{M} \rightarrow T^*\mathbf{M} \otimes \mathbf{F}_L \otimes \mathbf{F}_L^\star$, respectively, suitable for playing the role of gauge fields.

As for the last two terms in the above bundle expansion, we note that the identity is a distinguished section of $\mathbf{U}^\star \otimes \mathbf{U} \cong \text{End } \mathbf{U}$. A similar observation holds for $\overline{\mathbf{U}} \otimes \overline{\mathbf{U}}^\star \cong \text{End } \overline{\mathbf{U}}$. If we restrict our consideration, in these sectors, to sections which are proportional to the identity, we obtain sections $\mathbf{M} \rightarrow \overline{\mathbf{F}}_R \otimes \mathbf{F}_L$ and $\mathbf{M} \rightarrow \overline{\mathbf{F}}_L \otimes \mathbf{F}_R$, suitable for the role of *Higgs*

fields. On the other hand, nothing forbids to consider a larger class of fields; that would complicate the matter considerably, but could be intriguing in consideration of the still elusive properties of the recently detected Higgs bosons. Furthermore, one may examine the expansions of $Y \otimes Y^\star$, $Y \otimes \bar{Y}^\star$ and their conjugate bundles. Most sectors are, up to natural isomorphisms, the same already picked, but we do get some new ones. In particular, we get $F_R \otimes F_R^\star$ and $F_L \otimes F_L^\star$, suitable for describing *ghost fields*. So, our scheme for generating a gauge theory, though somewhat restricted with respect to full generality, still leaves the room for various kinds of natural extensions.

Eventually, recalling that the Hermitian structures of F_R and F_L determine (§3.1) Lie algebra sub-bundles $\mathfrak{L}_R \subset F_R \otimes F_R^\star$ and $\mathfrak{L}_L \subset F_L \otimes F_L^\star$, we make the further assumptions that the targets of gauge and ghost fields are restricted accordingly, so that the field list in “momentum representation” is:

- the *matter field* $\Psi \equiv (\Psi_R, \Psi_L) : P \rightarrow Y$, $P \equiv T^*M$;
- *gauge fields* $W_R : P \rightarrow P \otimes \mathfrak{L}_R$ and $W_L : P \rightarrow P \otimes \mathfrak{L}_L$;
- *Higgs and anti-Higgs fields*, $\phi : P \rightarrow F_L \otimes F_R^\star$ and $\phi^\dagger \equiv \bar{\phi} : P \rightarrow F_R \otimes F_L^\star$;
- *ghosts* $\omega_R : P \rightarrow \mathfrak{L}_R$ and $\omega_L : P \rightarrow \mathfrak{L}_L$;
- *anti-ghosts* $\varpi_R : P \rightarrow \mathfrak{L}_R^*$ and $\varpi_L : P \rightarrow \mathfrak{L}_L^*$.

We remark that ghosts and anti-ghosts are considered as independent fields, though the geometric structure would allow a precise relation between them; on the other hand, ϕ and $\bar{\phi}$ are mutually conjugated (could be independent fields as well in some extended theory).

In previous papers [11, 13] I showed in some detail how the above scheme fits electroweak theory, with the settings $F_L \equiv I$ (the *isospin bundle*) and $F_R \equiv \wedge^2 I$.

1.3 Symmetry breaking

Our general picture of a gauge theory has to be completed by a description of symmetry breaking. The “vacuum value” of the Higgs field is assumed to be a section

$$\mathcal{H}_0 : M \rightarrow F_L \otimes F_R^\star ,$$

which determines a splitting

$$F_L = F_R' \oplus_M F_R^\perp , \quad F_R' \equiv \mathcal{H}_0(F_R) .$$

We’ll assume \mathcal{H}_0 to be of maximal rank $\dim F_R$, so that it determines an isomorphism $F_R \rightarrow F_R' \subset F_L$. Then the matter field can be decomposed as

$$\Psi \equiv (\Psi_R, \Psi_L) = (\Psi_R, \Psi_R', \Psi_R^\perp) \equiv (\psi, \nu) ,$$

where

$$\psi \equiv (\Psi_R, \Psi_R') : M \rightarrow (F_R \otimes U) \oplus (F_R' \otimes \bar{U}^\star) \cong F_R \otimes W ,$$

$$\nu \equiv \Psi_R^\perp : M \rightarrow F_R' \otimes \bar{U}^\star \subset F_L \otimes \bar{U}^\star .$$

The field \mathcal{H}_0 can be regarded as an added feature of the underlying classical geometric structure. It’s natural to assume it has the further property of being conformally isometric, namely

$$h_L \circ (\bar{\mathcal{H}}_0, \mathcal{H}_0) = \frac{\mu^2}{\dim F_R} h_R , \quad \mu \in \mathbb{R} ,$$

where $h_L : M \rightarrow \overline{F}_L^\star \otimes F_L^\star$ and $h_R : M \rightarrow \overline{F}_R^\star \otimes F_R^\star$ denote the Hermitian metrics of F_R and F_L . This condition implies $\langle \bar{\mathcal{H}}_0, \mathcal{H}_0 \rangle = \mu^2$, so that \mathcal{H}_0 is a minimum of the ‘‘Higgs potential’’

$$\lambda (2\mu^2 \langle \bar{\phi}, \phi \rangle - \langle \bar{\phi}, \phi \rangle^2), \quad \lambda \in \mathbb{R}^+,$$

where $\langle \bar{\phi}, \phi \rangle \equiv \langle h_L \circ (\bar{\phi}, \phi), h_R^\# \rangle$ denotes the scalar obtained by contraction of $\bar{\phi} \otimes \phi$ via the Hermitian structure.

The \mathcal{H}_0 -splitting of F_L , together with the metric h_L , yields a splitting $F_R^\star = F_R'^\star \oplus F_R^{\perp\star}$, so that

$$\begin{aligned} \mathfrak{L}_L \subset \text{End } F_L &= (F_R' \oplus F_R^\perp) \otimes (F_R'^\star \oplus F_R^{\perp\star}) = \\ &= (F_R' \otimes F_R'^\star) \oplus (F_R^\perp \otimes F_R'^\star) \oplus (F_R' \otimes F_R^{\perp\star}) \oplus (F_R^\perp \otimes F_R^{\perp\star}). \end{aligned}$$

Now consider the decomposition of any $\xi \in \mathfrak{L}_L$ as

$$\xi = \xi' + \xi^+ + \xi^- + \xi^\perp \in \mathfrak{L}_R' \oplus \mathfrak{L}_R^+ \oplus \mathfrak{L}_R^- \oplus \mathfrak{L}_R^\perp,$$

where $\mathfrak{L}_R' \subset F_R' \otimes F_R'^\star$, $\mathfrak{L}_R^\perp \subset F_R^\perp \otimes F_R^{\perp\star}$, and

$$\mathfrak{L}_R^+ \equiv F_R^\perp \otimes F_R'^\star, \quad \mathfrak{L}_R^- \equiv F_R' \otimes F_R^{\perp\star}.$$

Since \mathfrak{L}_R^+ and \mathfrak{L}_R^- are anti-isomorphic by Hermitian adjunction, and any $\xi \in \mathfrak{L}_L$ (being anti-Hermitian) fulfills $(\xi^-)^\dagger = -\xi^+$, eventually we get a splitting

$$\mathfrak{L}_L \cong \mathfrak{L}_R' \oplus \mathfrak{L}_R^+ \oplus \mathfrak{L}_R^\perp \cong \mathfrak{L}_R' \oplus \mathfrak{L}_R^- \oplus \mathfrak{L}_R^\perp,$$

and, accordingly, a decomposition of the gauge, ghost and anti-ghost fields in the left sector.

2 Quantum states and interactions

2.1 Quantum states as generalised semi-densities

Let $Z \rightarrow X$ be a finite-dimensional complex vector bundle, $\dim_{\mathbb{R}} X = m$. Assume that X is *orientable*, and choose a positive semi-vector bundle $\mathbb{V} \equiv \mathbb{V}X \equiv (\wedge^m T^*X)^\perp$. A section $X \rightarrow \mathbb{V}^{-1/2} \otimes Z$ is called a *Z-valued semi-density*. The vector space of all such sections which are smooth and have compact support is denoted as $\mathcal{P}_0(X, Z)$. The dual space of $\mathcal{P}_0(X, Z^\star)$ in the standard topology [24] is indicated as $\mathcal{P}(X, Z)$ and called the space of *generalised semi-densities* of $Z \rightarrow X$. In particular, a sufficiently regular ordinary section $\theta : X \rightarrow \mathbb{V}^{-1/2} \otimes Z$ can be seen as an element in $\mathcal{P}(X, Z)$ via the rule $\langle \theta, \sigma \rangle := \int_X \langle \theta(x), \sigma(x) \rangle$, $\sigma \in \mathcal{P}_0(X, Z^\star)$.

Semi-densities have a special status among all kinds of generalised sections, since there is a natural inclusion $\mathcal{P}_0(X, Z) \subset \mathcal{P}(X, Z)$. Furthermore, if a fibered Hermitian structure of $Z \rightarrow X$ is assigned then one has the space $\mathcal{L}^2(X, Z)$ of all ordinary semi-densities θ such that $\langle \theta^\dagger, \theta \rangle < \infty$. The quotient $\mathcal{H}(X, Z) = \mathcal{L}^2(X, Z)/\mathbf{0}$ is then a Hilbert space (here $\mathbf{0} \subset \mathcal{L}^2(X, Z)$ denotes the subspace of all almost-everywhere vanishing sections), and we get a so-called *rigged Hilbert space* [1]

$$\mathcal{P}_0(X, Z) \subset \mathcal{H}(X, Z) \subset \mathcal{P}(X, Z).$$

Elements in $\mathcal{P}(X, Z) \setminus \mathcal{H}(X, Z)$ can then be identified with the (*non-normalisable*) *generalised states* of the common terminology.

Let $\delta[x]$ be the *Dirac density* on \mathbf{X} with support $\{x\}$, $x \in \mathbf{X}$. A generalised semi-density is said to be of *Dirac type* if it is of the form $\delta[x] \otimes v \in \mathcal{P}(\mathbf{X}, \mathbf{Z})$ with $v : \mathbf{X} \rightarrow \mathbb{V}^{1/2} \otimes \mathbf{Z}$. If (\mathbf{b}_α) is a frame of $\mathbf{Z} \rightarrow \mathbf{X}$ then we set

$$|x\rangle \otimes \mathbf{b}_\alpha(x) \leftrightarrow \mathbf{B}_{x,\alpha} \equiv \delta[x] \otimes \eta^{-1/2} \otimes \mathbf{b}_\alpha(x) ,$$

and call the set $(\mathbf{B}_{x,\alpha}) \subset \mathcal{P}^1$ a *generalised basis*. Accordingly we introduce a handy “generalised index” notation. We write $\mathbf{B}^{x,\alpha} \equiv \delta[x] \otimes \eta^{-1/2} \otimes \mathbf{b}^\alpha(x)$, where (\mathbf{b}^α) is the dual classical frame. Though contraction of two distributions is not defined in general, a straightforward extension of the discrete-space operation yields

$$\langle \mathbf{B}^{x',\alpha'}, \mathbf{B}_{x,\alpha} \rangle = \delta_{x,\alpha}^{x',\alpha'} \eta(x) .$$

which is consistent with “index summation” in a generalised sense: if $f \in \mathcal{P}_o(\mathbf{X}, \mathbf{Z})$ and $\lambda \in \mathcal{P}_o(\mathbf{X}, \mathbf{Z}^\star)$ are test semi-densities, then we write

$$f^{x,\alpha} \equiv f^\alpha(x) \equiv \langle \mathbf{B}^{x,\alpha}, f \rangle , \quad \lambda_{x,\alpha} \equiv \lambda_\alpha(x) \equiv \langle \lambda, \mathbf{B}_{x,\alpha} \rangle ,$$

$$\langle \lambda, f \rangle \equiv \lambda_{x',\alpha'} f^{x,\alpha} \langle \mathbf{B}^{x',\alpha'}, \mathbf{B}_{x,\alpha} \rangle \equiv \int_{\mathbf{X}} \lambda_\alpha(x) f^\alpha(x) \eta(x) ,$$

namely we interpret index summation with respect to the continuous variable x as integration (provided by the chosen volume form). This formalism can be extended to the contraction of two generalised semi-densities whenever it makes sense.

Next we set $\mathcal{Z}^1 \equiv \mathcal{P}(\mathbf{X}, \mathbf{Z})$, which is our template for the space of states of one particle of some type. The associated “ n -particle state” space \mathcal{Z}^n is defined to be either the symmetrised tensor product $\vee^n \mathcal{Z}^1$ (*bosons*) or the anti-symmetrised tensor product $\wedge^n \mathcal{Z}^1$ (*fermions*). The “multi-particle state” space is defined to be $\mathcal{Z} \equiv \bigoplus_{n=0}^\infty \mathcal{Z}^n$ (constituted by finite sums with arbitrarily many terms). Similarly we set $\mathcal{Z}^{\star 1} \equiv \mathcal{P}(\mathbf{X}, \mathbf{Z}^\star)$ and define the “dual” multi-particle space to be $\mathcal{Z}^\star \equiv \bigoplus_{n=0}^\infty \mathcal{Z}^{\star n}$.

A general theory of quantum particles has several particle types. Correspondingly, one considers several multi-particle state spaces (or “sectors”) \mathcal{Z}' , \mathcal{Z}'' , \mathcal{Z}''' etc. The total state space is now defined to be

$$\mathcal{V} := \mathcal{Z}' \otimes \mathcal{Z}'' \otimes \mathcal{Z}''' \otimes \dots = \bigoplus_{n=0}^\infty \mathcal{V}^n$$

where \mathcal{V}^n , constituted of all elements of tensor rank n , is the space of all states on n particles of any type. We observe that if \mathcal{X} and \mathcal{Y} are any two vector spaces, then their antisymmetric tensor algebras fulfill the isomorphisms

$$\wedge^p(\mathcal{X} \oplus \mathcal{Y}) \cong \bigoplus_{h=0}^p (\wedge^{p-h} \mathcal{X}) \otimes (\wedge^h \mathcal{Y}) , \quad (\wedge \mathcal{X}) \otimes (\wedge \mathcal{Y}) \cong \wedge(\mathcal{X} \oplus \mathcal{Y}) .$$

Hence all fermionic sectors can be described by a unique overall antisymmetrised tensor algebra. A similar observation holds true for the bosonic sectors, while we regard mutual ordering of fermionic and bosonic sectors as inessential. Similarly one constructs a “dual” space $\mathcal{V}^\star := \mathcal{Z}^{\star'} \otimes \mathcal{Z}^{\star''} \otimes \mathcal{Z}^{\star'''} \dots = \bigoplus_{n=0}^\infty \mathcal{V}^{\star n}$.

2.2 Quantum bundles, detectors and free-particle states

In this paper we use the term *quantum bundle* to mean a vector bundle over spacetime whose fibers are distributional spaces [8, 13]. The underlying “classical” (i.e. finite-dimensional)

geometric structure is that of a 2-fibered bundle, and the infinite-dimensional smooth structure is conveniently treated in terms of Frölicher's notion of smoothness, or *F-smoothness* [14, 15, 18, 16, 2, 20]. The F-smooth geometry of distributional bundles and quantum connections has been studied in a previous paper [7].

Let (M, g) be Einstein's spacetime. Taking the speed of light and the Planck constant into account, the covariant form of a particle's 4-momentum is valued into $P_m \subset P \cong T^*M$, the sub-bundle over M of future 'mass -shells' corresponding to mass $m \in \{0\} \cup \mathbb{L}^{-1}$ (\mathbb{L} is the semi-space of length units). Let now $Z \rightarrow P_m$ be a vector bundle (representing the 'internal degrees of freedom' of the considered particle type). The constructions of §2.1 at each $x \in M$, with the generic manifold X replaced by $(P_m)_x$, yield spaces Z_x^1 , and the fibered set $Z^1 := \bigsqcup_{x \in M} Z_x^1$ turns out to have a natural F-smooth vector-bundle structure over M . The multi-particle bundle $Z := \bigoplus_n Z^n \rightarrow M$, $n \in \{0\} \cup \mathbb{N}$, can also be straightforwardly constructed.

Let $g^\#$ be the "contravariant" metric induced on the fibers of $P \equiv T^*M$. When an observer (a congruence of timelike curves) is considered then one has the orthogonal splitting $P = P_\parallel \oplus_T P_\perp$, and the volume form η_\perp on the fibers of $P_\perp \rightarrow M$. The orthogonal projection $P \rightarrow P_\perp$ yields a distinguished diffeomorphism $P_m \leftrightarrow P_\perp$ for each m . The pull-back of η_\perp is then a volume form on the fibers of P_m , which is denoted for simplicity by the same symbol.

It will be convenient to use the "spatial part" p_\perp of the 4-momentum p as a label, that is a generalised index for quantum states. For each $p \in P_m$ let $\delta_m[p]$ the Dirac density with support $\{p\}$ on the same fiber of P_m , and $\delta(y_\perp - p_\perp)$ the generalised function characterised by $\delta_m[p](y) = \delta(y_\perp - p_\perp) d^3y$, where we are using linear coordinates $(y_\lambda) \equiv (y_0, y_1, y_2, y_3) \equiv (y_0, y_\perp)$ in the fibers of P . Now consider the section $P_m \rightarrow \mathcal{P}(P_m, \mathbb{C}) : p \mapsto X_p$ defined as follows; for each $p \in P_m$ we can regard X_p as a generalised function of the variable y_\perp , with the expression

$$X_p(y) := l^{-3/2} \delta(y_\perp - p_\perp) \sqrt{d^3y}.$$

Here l is a constant length needed in order to get an unscaled semi-density (compare with the usual "box quantization" argument). Eventually, we get the distinguished isomorphism $Z^1 \leftrightarrow \mathcal{P}(P_m, Z)$ which is determined by the correspondence $|z\rangle \leftrightarrow X_p \otimes z$, $z \in Z_p$.

We can develop our arguments by assuming a weaker structure than a congruence of space-time submanifolds, namely a unique timelike submanifold $T \subset M$, which we call a *detector*. Through a natural construction exploiting the exponential map, we also get a timelike congruence in a neighbourhood of T . Actually this setting suffices for reproducing, in terms of generalised semi-densities, essentially the standard momentum space formalism, which can be seen as a sort of a complicated 'clock' carried by the detector [8, 13].

We obtain a *generalised frame of free one-particle states* along T as follows. First, at some arbitrarily fixed event $t_0 \in T \subset M$ we choose a frame (b_α) of $Z \rightarrow (P_m)_{t_0}$. Then the family of generalised semi-densities $B_{p\alpha}(t_0) \equiv X_p \otimes b_\alpha \in \mathcal{P}(P_m, Z)_{t_0}$ is a generalised frame at t_0 . We transport $B_{p\alpha}$ along T by means of Fermi transport [10, 13] for the spacetime and spinor factors,² and, for the remaining factors, by means of parallel transport relatively to a suitable connection of Z which will have to be assumed (see also §3.4). We write

$$B_{p\alpha} : T \rightarrow \mathcal{P}(P_m, Z)_T : t \mapsto B_{p\alpha}(t) = X_{p(t)} \otimes b_\alpha,$$

where $p : T \rightarrow P_m : t \mapsto p(t)$ is Fermi-transported. This yields a trivialization

$$\mathcal{P}(P_m, Z)_T \cong T \times \mathcal{P}(P_m, Z)_{t_0},$$

²Fermi and parallel transport coincide if the detector is inertial.

which can be seen as determined by a suitable connection called the *free-particle connection*. Eventually, the above arguments can be naturally extended to multi-particle bundles and states. When several particle types are considered, we get a trivialization $\mathcal{V}_T \cong T \times \mathcal{Q}$ of the total quantum state bundle so that $\mathcal{Q} \equiv \mathcal{V}_{t_0}$ can be identified with the space of all asymptotical quantum states. The quantum interaction can be constructed, assembling the classical interaction with a distinguished quantum ingredient, as a modification of that parallel transport. The free-particle trivialization preserves particle type and number by construction, while the interaction doesn't.

2.3 Quantum interactions

Quantum interactions are described by a section

$$-i dt \otimes \mathfrak{H} : T \rightarrow T^*T \otimes \text{End}(\mathcal{V}) ,$$

where the scaled function t is the detector's *proper time*. A *quantum history* is defined to be a section $T \rightarrow \mathcal{V}_T$, which we conveniently regard as a map $\psi : T \rightarrow \mathcal{Q} \equiv \mathcal{V}_{t_0}$, obeying the law $\psi(t) = \mathcal{U}_{t_0}(t) \psi(t_0)$, where $\mathcal{U}_{t_0} : T \rightarrow \text{End}(\mathcal{V}_{t_0})$ is determined by the differential equation

$$\frac{d}{dt} \mathcal{U}_{t_0}(t) = -i \mathfrak{H}(t) \circ \mathcal{U}_{t_0}(t) , \quad \mathcal{U}_{t_0}(t_0) = \mathbb{1}_{\mathcal{Q}} .$$

The free-particle connection yields the trivialisation $\mathcal{V}_T \rightarrow T \times \mathcal{Q}$; we can see the interaction as a tensor field which modifies that connection and determines a new quantum connection of the functional bundle $\mathcal{V}_T \rightarrow T$. Or, the interaction can be seen as a 1-form on T valued into the endomorphisms of the fixed space \mathcal{Q} .

The time-dependent endomorphism \mathfrak{H} is dictated, in an essentially elementary way, by the underlying “classical structure”, while the problem of determining \mathcal{U} is on a different footing. In perturbative theories one starts from the *Dyson series*, which provides a formal solution of the above differential equation, and tries to extract meaningful physical results from the study of the *scattering operator*

$$\mathcal{S} := \lim_{\substack{t \rightarrow +\infty \\ t_0 \rightarrow -\infty}} \mathcal{U}_{t_0}(t) ,$$

which, intuitively, relates asymptotical states of ‘incoming’ and ‘outgoing’ free particles interacting in a small spacetime region.

Essentially, \mathfrak{H} arises as the tensor product of the classical interaction and a certain semi-density on particle momenta (the “quantum ingredient” of the interaction). Consider masses $m', m'', \dots, m^{(r)}$ and let the shorthand

$$\mathbf{P}_{\times} := \mathbf{P}_{m'} \times_M \mathbf{P}_{m''} \times_M \dots \times_M \mathbf{P}_{m^{(r)}} \rightarrow M$$

denote the bundle of r particle momenta corresponding to these masses. Let $\delta^{(r)}$ be the fiberwise generalised function³ on \mathbf{P}_{\times} characterised by

$$\langle \delta^{(r)} , f \rangle = \iint \check{f}(y'_\perp, y''_\perp, \dots, y^{(r-1)}_\perp, -\sum_{i=1}^{r-1} y^{(i)}_\perp) d^3 y' d^3 y'' \dots d^3 y^{(r-1)}$$

for any test density $f = \check{f} d^3 y' \otimes d^3 y'' \otimes \dots \otimes d^3 y^{(r)}$, namely $\delta^{(r)} \equiv \delta(y'_\perp + y''_\perp + \dots + y^{(r)}_\perp)$ in standard notation. Recalling that on the fibers of $\mathbf{P}_m \rightarrow M$ we have the natural *Leray form*, which

³Namely $\delta^{(r)}$ is a section of a distributional bundle [7] over M . For the sake of brevity, in this paper we are not going to explicitly denote all involved spaces.

can be then written as $\omega_m(p) = (2p_0)^{-1} \eta_\perp(p)$, $p \in \mathbf{P}_m$, where $p_0 \equiv E_m(p_\perp) = (m^2 + p_\perp^2)^{1/2}$, we introduce the generalized half-density

$$\underline{\Lambda}^{(r)} := \delta^{(r)} \sqrt{\omega_{m'}} \otimes \cdots \otimes \sqrt{\omega_{m^{(r)}}} = \frac{\delta(y'_\perp + \cdots + y^{(r)}_\perp)}{\sqrt{2^r y'_0 \cdots y_0^{(r)}}} \sqrt{d^3 y'} \otimes \cdots \otimes \sqrt{d^3 y^{(r)}}.$$

By multiplying $\underline{\Lambda}^{(r)}$ by certain phase factors we introduce a modified generalized half-density $\Lambda^{(r)}$, which can be expressed in the generalised index notation as⁴

$$\begin{aligned} \Lambda^{(r)} &= \Lambda^{p'p''p'''\cdots} X_{p'} \otimes X_{p''} \otimes X_{p'''} \otimes \cdots, \\ \Lambda^{p'p''p'''\cdots} &\equiv l^{2r} (2^r p'_0 p''_0 p'''_0 \cdots)^{-1/2} e^{-i(p'_0 + p''_0 + p'''_0 + \cdots)\mathfrak{t}} \delta(p'_\perp + p''_\perp + p'''_\perp + \cdots). \end{aligned}$$

We consider a special rule for lowering and rising indices in $\Lambda^{(r)}$, so obtaining tensors of different index types, associated with $\Lambda^{(r)}$, as in finite-dimensional tensor algebra. This rule (which can be seen as related to a Hermitian structure) prescribes that moving an index $p^{(i)}$ you change the sign of $p_0^{(i)}$ in the exponential and the sign of $p_\perp^{(i)}$ in the delta-function, so that

$$\Lambda_{p'p''p'''\cdots} = l^{2r} \frac{e^{-i(-p'_0 + p''_0 + p'''_0 + \cdots)\mathfrak{t}}}{(2^r p'_0 p''_0 p'''_0 \cdots)^{1/2}} \delta(-p'_\perp + p''_\perp + p'''_\perp + \cdots)$$

and the like.

Now the tensor field $\ell^{(r)}$ describing the classical interaction of r particles (i.e. the interaction lagrangian) yields as many index types as $\Lambda^{(r)}$, and the corresponding index types from $\ell^{(r)} \otimes \Lambda^{(r)}$ generate all the various pieces of the quantum interaction, namely the various terms in \mathfrak{H} . A term with s low indices describes, via a generalised analogue of an elementary algebraic mechanism, the absorption of s particles and the emission of $r - s$. Propagators and all particle interactions in gauge theories can be indeed recovered from the above ideas (see previous papers [8, 13] for details).

3 Gauge freedom

3.1 Classical gauge freedom

If the “matter field” of a classical theory is a section of the bundle $\mathbf{E} \rightarrow \mathbf{M}$, then the classical “gauge field” is a connection of that bundle, namely a section $\mathbf{E} \rightarrow \mathbf{J}\mathbf{E}$ of the 1-jet bundle. If $\mathbf{E} \rightarrow \mathbf{M}$ is a vector bundle then, in particular, we consider linear connections, which can be seen as sections $\mathbf{M} \rightarrow \mathbf{\Gamma}$ where $\mathbf{\Gamma} \subset \mathbf{J}\mathbf{E} \otimes_{\mathbf{M}} \mathbf{E}^*$ is the sub-bundle projecting over the identity $\mathbb{1}_{\mathbf{E}}$. This is an affine bundle, with “derived” vector bundle $D\mathbf{\Gamma} = T^*\mathbf{M} \otimes_{\mathbf{M}} \text{End } \mathbf{E}$ (the bundle of “differences of linear connections”).

The fibers of the vector bundle $\text{End } \mathbf{E} \rightarrow \mathbf{M}$ are constituted by all linear endomorphisms of the respective fibers of \mathbf{E} , and are naturally Lie algebras via by the ordinary commutator. In fact, this is the Lie algebra bundle of the group bundle $\text{Aut } \mathbf{E} \rightarrow \mathbf{M}$ of all automorphisms of the fibers of \mathbf{E} . However, $\mathbf{E} \rightarrow \mathbf{M}$ is usually endowed with some fibered geometric structure, which selects the (“internal” symmetry) Lie-group subbundle $\mathbf{G} \rightarrow \mathbf{M}$ of all automorphisms preserving it; the fibers of \mathbf{G} are isomorphic Lie groups, though distinguished isomorphisms among them don’t exist in general.⁵ The Lie algebra bundle of \mathbf{G} is a sub-bundle $\mathfrak{L} \subset \text{End } \mathbf{E}$.

⁴Generalised index summation is interpreted as integration (§2.1) via the volume form η_\perp .

⁵In order to deal with a *fixed group* one can exploit the notion of a principal bundle.

By ordinary restriction we obtain the affine sub-bundle $\Gamma_G \subset \Gamma$, with derived vector bundle $D\Gamma_G = T^*\mathbf{M} \otimes_{\mathbf{M}} \mathfrak{L}$; sections $\mathbf{M} \rightarrow \Gamma_G$ characterise those linear connections which preserve the fiber geometric structure (i.e. make it covariantly constant). Hence the difference of any two such connections is \mathfrak{L} -valued.

Connections can be locally described as tensor fields by choosing a gauge, namely a local “flat” connection γ_0 . In fact the difference $\alpha \equiv \gamma - \gamma_0 : \mathbf{E} \rightarrow T^*\mathbf{M} \otimes_{\mathbf{E}} \mathbf{V}\mathbf{E}$ determines an arbitrary connection γ , and, since we are dealing with linear connections, we can write

$$\alpha : \mathbf{M} \rightarrow T^*\mathbf{M} \otimes_{\mathbf{M}} \text{End } \mathbf{E} \equiv T^*\mathbf{M} \otimes_{\mathbf{M}} \mathbf{E} \otimes_{\mathbf{M}} \mathbf{E}^* .$$

The curvature tensor $\rho \equiv [\gamma, \gamma]$ (Frölicher-Nijenhuis bracket) can be expressed in terms of α as $2[\gamma_0, \alpha] + [\alpha, \alpha]$.

The γ_0 -constant local sections of $\mathbf{E} \rightarrow \mathbf{M}$ determine a trivialization of \mathbf{E} over any sufficiently small open subset of \mathbf{M} . Thus one also has γ_0 -constant local frames. Conversely, the assignment of a local frame determines a flat connection γ_0 by the condition that its coefficients vanish in that frame.

A (local) *gauge transformation* is defined to be a section $K : \mathbf{M} \rightarrow \mathbf{G}$. Together with its transposed inverse \overleftarrow{K}^* , a fibered automorphism of $\mathbf{E}^* \rightarrow \mathbf{M}$, it determines a fibered automorphism of the whole tensor algebra of $\mathbf{E} \times_{\mathbf{M}} \mathbf{E}^*$. Moreover, K transforms the family of γ_0 -constant sections to a new family of sections, which determines a new flat connection $\gamma'_0 = \gamma_0 + (\nabla[\gamma_0]K) \rfloor \overleftarrow{K}$. In particular, if $\nabla[\gamma_0]K = 0$ then $\gamma'_0 = \gamma_0$, namely the two families of covariantly constant sections coincide.⁶

Let now $\gamma = \alpha - \gamma_0 = \alpha' - \gamma'_0$ be a fixed connection; we get $\alpha' - \alpha = (\nabla[\gamma_0]K) \rfloor \overleftarrow{K}$, namely $\alpha, \alpha' : \mathbf{M} \rightarrow T^*\mathbf{M} \otimes_{\mathbf{M}} \mathfrak{L}$ represent the same connection whenever their difference is of that type. This implies that any scalars formed from covariant derivatives of tensor fields and from the curvature tensor of γ are invariant under the replacement $\alpha \leftrightarrow \alpha'$. So we can look at the notion of gauge freedom as follows: if we insist in describing gauge fields (i.e. connections) in terms of tensor fields then we concede them too many “degrees of freedom”, which must be absorbed by taking a suitable quotient. The crucial point is that in quantum theory the fields must be sections of vector bundles.

Still in view of quantum theory we consider gauge fields in terms of momenta. We take a hint from the observation that a radiative electromagnetic field is usually assumed [19] to be of the form $F = k \wedge b$, with $k, b : \mathbf{M} \rightarrow T^*\mathbf{M}$ such that $k^\#$ is a geodesic null vector field and $g^\#(k, b) = 0$. While in curved spacetime we may not be able to find a *closed* such tensor field,⁷ it makes sense to use it as a template for our description of photons. The couple (k, b) constitutes then a *redundant* description, being a representative of an equivalence class characterising the e.m. potential. We can describe a more general gauge field by a couple (k, α) , with $\alpha : \mathbf{M} \rightarrow T^*\mathbf{M} \otimes_{\mathbf{M}} \mathfrak{L}$, such that $k^\# \rfloor \alpha = 0$. The physical meaning of the gauge field is encoded by its equivalence class, (k, α) and (k, α') being equivalent if their difference is of the kind $k \otimes \chi$ with $\chi : \mathbf{M} \rightarrow \mathfrak{L}$.

The equivalence class of (k, α) also uniquely determines the “curvature-like” tensor

$$\rho[k, \alpha] := i k \wedge \alpha + \alpha \bar{\wedge} \alpha ,$$

⁶In that case one uses to say that K is a “global” gauge transformation.

⁷Radiative e.m. fields in curved spacetime are usually dealt with by considering solutions of the Maxwell equations which approximate said type in the small wavelength limit.

where the notation $\alpha \bar{\wedge} \beta$ stands for exterior product of \mathfrak{L} -valued forms together with composition.⁸ In the quantum theory in momentum representation, the Lagrangian for the gauge field, which is expressed in terms of ρ , yields all self-interaction terms. The replacement $\alpha \rightarrow k \otimes \chi + \alpha$ does not affect any scattering matrix calculations. According to the usual quantisation procedure, this freedom can be exploited by adding to the Lagrangian density a suitable term (proportional to the squared divergence of α) which is not gauge-invariant, namely does not “pass to the quotient” when we deal with the above said equivalence classes, though it is a natural geometric object when α is seen as a tensor field. This modifies the gauge particle propagator in a way that does not affect point interactions.

3.2 Two-spinors and one-particle states in QED

Let $\mathbf{P}_m \rightarrow \mathbf{M}$ (§2.2) be the sub-bundle of $\mathbf{T}^*\mathbf{M}$ whose fibers are the mass-shells corresponding to mass $m \in \{0\} \cup \mathbb{L}^{-1}$. If $p \in (\mathbf{P}_m)_x$, $x \in \mathbf{M}$, then we have the *Dirac splitting*

$$\mathbf{W}_x = \mathbf{W}_p^+ \oplus \mathbf{W}_p^-, \quad \mathbf{W}_p^\pm := \text{Ker}(\gamma[p^\#] \mp m),$$

where $p^\# \equiv g^\#(p) \in \mathbb{L}^{-2} \otimes \mathbf{T}\mathbf{M}$ is the contravariant form of p . Thus we obtain 2-fibered bundles $\mathbf{W}_m^\pm \rightarrow \mathbf{P}_m \rightarrow \mathbf{M}$, where

$$\mathbf{W}_m^\pm := \bigsqcup_{p \in \mathbf{P}_m} \mathbf{W}_p^\pm \subset \mathbf{P}_m \times_{\mathbf{M}} \mathbf{W}.$$

We call \mathbf{W}_m^+ and $\overline{\mathbf{W}}_m^-$ the *electron bundle* and the *positron bundle*, respectively. If $(\zeta_A(p))$ is a 2-spinor frame such that $p^\# \propto \tau_0$ in the associated Pauli frame, then the *Dirac frame* $(u_A(p), v_B(p))$, defined by

$$u_1 \equiv \frac{1}{\sqrt{2}}(\zeta_1, \bar{\zeta}^1), \quad u_2 \equiv \frac{1}{\sqrt{2}}(\zeta_2, \bar{\zeta}^2), \quad v_1 \equiv \frac{1}{\sqrt{2}}(\zeta_1, -\bar{\zeta}^1), \quad v_2 \equiv \frac{1}{\sqrt{2}}(\zeta_2, -\bar{\zeta}^2),$$

is k-orthonormal and adapted to the Dirac splitting.

The splitting has an interesting two-spinor description [9]. If $\psi \equiv (u, \bar{\lambda}) \in \mathbf{W}$ then

$$\tau \equiv \frac{1}{\sqrt{2}|\langle \lambda, u \rangle|} (u \otimes \bar{u} + \lambda^\# \otimes \bar{\lambda}^\#) \in \mathbf{H}$$

is a unit future-pointing timelike vector. By a straightforward calculation one sees that $\gamma[\tau]\psi = \pm\psi$ if and only if $\langle \lambda, u \rangle \in \mathbb{R}^\pm$. Conversely, it can be proved that if $\tau' \in \mathbf{H}$ is such that $\gamma[\tau']\psi = \pm\psi$, then necessarily $\tau' = \tau$. In other terms, *internal states of free electrons and positrons carry the full information about their momenta*.

For a fixed a detector $\mathbf{T} \subset \mathbf{M}$, we use generalised electron and positron frames

$$\mathbf{A}_{pA} := e^{-ip_0 t} \mathbf{X}_p \otimes u_A(p) : \mathbf{T} \rightarrow \mathcal{D}(\mathbf{P}_m, \mathbf{W}_m^+),$$

$$\mathbf{C}_{pA'} := e^{-ip_0 t} \mathbf{X}_p \otimes \bar{v}_{A'}(p) : \mathbf{T} \rightarrow \mathcal{D}(\mathbf{P}_m, \overline{\mathbf{W}}_m^-),$$

where $p : \mathbf{T} \rightarrow \mathbf{P}_m$ is Fermi-transported.

We discuss real photon states in spacetime terms first. Consider the zero-mass subbbundle $\mathbf{P}_0 \subset \mathbf{T}^*\mathbf{M}$ of future null half-cones. We use the identification $\mathbf{H}^* \cong \mathbb{L} \otimes \mathbf{T}^*\mathbf{M}$ determined by the fixed tetrad. Let $\mathbf{H}' \subset \mathbf{P}_0 \times_{\mathbf{M}} \mathbf{H}^*$ be the sub-bundle over \mathbf{M} whose fiber over any $k \in (\mathbf{P}_0)_x$,

⁸In components, $(\alpha \bar{\wedge} \beta)_{ab}^i = c_{jk}^i \alpha_a^j \beta_b^k$ where $c_{jk}^i \equiv [l_j, l_k]^i$ are the “structure constants” in the chosen frame (l_i) of \mathfrak{L} .

$x \in M$, is the 3-dimensional real vector space $\mathbf{H}'_k := \{y \in \mathbf{H}^* : g^\#(k, y) = 0\}$. We have the real vector bundle $\mathbf{B}_\mathbb{R} \rightarrow \mathbf{P}_0$ whose fiber over any $k \in \mathbf{P}_0$ is the 2-dimensional quotient space \mathbf{H}'_k/k . It turns out that the (contravariant) spacetime metric ‘passes to the quotient’, so it naturally determines a negative metric g_B in the fibers of $\mathbf{B}_\mathbb{R} \rightarrow \mathbf{P}_0$, as well as a ‘Hodge’ isomorphism $*_B$ which can be characterised through the rule $*(k \wedge \beta) = -k \wedge (*_B \beta)$.

The complexified 2-fibered bundle $\mathbf{B} := \mathbb{C} \otimes \mathbf{B}_\mathbb{R} \rightarrow \mathbf{P}_0 \rightarrow M$ (the *optical bundle* [6, 21]) has the natural splitting

$$\mathbf{B} = \mathbf{B}^+ \oplus_{\mathbf{P}_0} \mathbf{B}^- ,$$

where the fibers of $\mathbf{B}^\pm \rightarrow \mathbf{P}_0$ are complex 1-dimensional g_B -null subspaces defined to be the eigenspaces of $-i*_B$ with eigenvalues ± 1 (*self-dual* and *anti-self-dual* subspaces). Restricting these bundles to the detector’s world line $T \subset M$ then we can identify $\mathbf{B}_\mathbb{R} \rightarrow \mathbf{P}_0 \rightarrow T$ with $\mathbf{H}' \cap \mathbf{H}^{*\perp} \rightarrow \mathbf{P}_0 \rightarrow T$ (‘radiation gauge’). For any $k \in (\mathbf{P}_0)_x$, $x \in M$, let (τ^λ) be a Pauli basis of \mathbf{H} at x such that τ_0 is tangent to T and $k^\# \propto \tau_0 + \tau_3$; setting

$$(\mathbf{b}_+, \mathbf{b}_-) \equiv (\mathbf{b}_1, \mathbf{b}_2) := \left(\frac{1}{\sqrt{2}} (\tau^1 + i\tau^2), \frac{1}{\sqrt{2}} (\tau^1 - i\tau^2) \right) \subset \mathbb{C} \otimes \mathbf{H}' \cap \mathbf{H}^{*\perp} ,$$

$$\mathbf{B}_{kQ} := e^{-ik_0 t} \mathbf{X}_k \otimes \mathbf{b}_Q(k) , \quad k \in \mathbf{P}_0, \quad Q = 1, 2 ,$$

one gets, by Fermi transport, a generalised frame $\{\mathbf{B}_{kQ}\}$ of the quantum bundle $\mathcal{P}_M(\mathbf{P}_0, \mathbf{B}) \rightarrow M$. This frame is adapted to the self-dual/anti-self-dual splitting.

Remark. While the photon’s physical meaning is encoded in the 2-form $k \wedge \beta$ (§3.1), the radiation gauge determines k and β separately.

We now observe (§1.1) that an element $\beta \in \mathbb{C} \otimes \mathbf{H}^* = \mathbf{U}^\star \otimes \overline{\mathbf{U}}^\star$ is null if and only if it is decomposable (i.e. a monomial), while future-pointing real null elements are of the type $k = \kappa \otimes \bar{\kappa}$. It’s not difficult to check that $\beta \in \mathbf{B}_k^\pm$ iff $\beta = \kappa \otimes \bar{\lambda}$ and $\beta = \lambda \otimes \bar{\kappa}$, respectively. On the other hand, sums of the type $\kappa \otimes \bar{\lambda} + \mu \otimes \bar{\nu}$ span the whole $\mathbb{C} \otimes \mathbf{H}^*$, which is also spanned by *virtual photons*. For the latter we can enlarge the frame $(\mathbf{b}_+, \mathbf{b}_-)$ by including, for example, $\mathbf{b}_0 \equiv \tau^0$ and $\mathbf{b}_3 \equiv \frac{1}{\sqrt{2}} (\tau^0 - \tau^3)$, respectively related to ‘scalar’ and ‘longitudinal’ modes.

3.3 QED interactions and gauge freedom in terms of two-spinors

In electrodynamics, the algebraic part of the point interaction can be described as the tensor field $\ell_{\text{int}} : M \rightarrow \overline{\mathbf{W}}^\star \otimes \mathbf{H} \otimes \mathbf{W}^\star$ defined by

$$\ell_{\text{int}}(\bar{\phi}, A, \psi) := -e \langle \bar{\phi}, \gamma[A^\#] \psi \rangle ,$$

where e is the positron’s charge. Index moving in the fibers is determined by Dirac adjunction⁹ in \mathbf{W} and by the Lorentz metric in \mathbf{H} . By using each factor in ℓ_{int} either as absorption or as emission we obtain the eight point interactions of QED, represented by the diagrams



(time flows upwards). In two-spinor terms, if $\phi = (v, \bar{\mu})$ and $\psi = (u, \bar{\lambda})$ we get

$$\langle \bar{\phi}, \gamma[r \otimes \bar{s}] \psi \rangle = \sqrt{2} (\langle \mu, r \rangle \langle \bar{\lambda}, \bar{s} \rangle + \varepsilon(u, r) \varepsilon(\bar{v}, \bar{s})) \equiv \sqrt{2} g(r \otimes \bar{s}, u \otimes \bar{v} + \mu^\# \otimes \bar{\lambda}^\#) ,$$

⁹We do *not* consider different index positions obtained through some positive Hermitian metric: that would be an extra structure, equivalent to the assignment of an observer.

hence in general $\langle \bar{\phi}, \gamma[A^\#]\psi \rangle = \sqrt{2} g(A^\#, u \otimes \bar{v} + \mu^\# \otimes \bar{\lambda}^\#)$. The kernel of the map

$$\langle \bar{\phi}, \gamma[\cdot]\psi \rangle : \mathbb{C} \otimes \mathbf{H}^* \rightarrow \mathbb{C} : A \mapsto \langle \bar{\phi}, \gamma[A^\#]\psi \rangle$$

is then constituted by all covectors orthogonal to $u \otimes \bar{v} + \mu^\# \otimes \bar{\lambda}^\# \in \mathbf{U} \otimes \bar{\mathbf{U}} = \mathbb{C} \otimes \mathbf{H}$.

In particular we observe that setting

$$k \equiv \frac{m}{\sqrt{2}} \left(\frac{(u \otimes \bar{u})^\flat + \lambda \otimes \bar{\lambda}}{|\langle \lambda, u \rangle|} \pm \frac{(v \otimes \bar{v})^\flat + \mu \otimes \bar{\mu}}{|\langle \mu, v \rangle|} \right)$$

by straightforward 2-spinor algebra calculations one obtains

$$\frac{1}{m} \langle \bar{\phi}, \gamma[k^\#]\psi \rangle = \langle \mu, u \rangle \left(\frac{\langle \bar{\lambda}, \bar{u} \rangle}{|\langle \lambda, u \rangle|} \pm \frac{\langle \bar{\mu}, \bar{v} \rangle}{|\langle \mu, v \rangle|} \right) + \langle \bar{\lambda}, \bar{v} \rangle \left(\frac{\langle \lambda, u \rangle}{|\langle \lambda, u \rangle|} \pm \frac{\langle \mu, v \rangle}{|\langle \mu, v \rangle|} \right).$$

It's easy to check [9] that the condition $\psi \equiv (u, \bar{\lambda}) \in \mathbf{W}^\pm$ can be expressed, in 2-spinor terms, as $\langle \lambda, u \rangle \in \mathbb{R}^\pm$. Hence the above expression vanishes when the minus sign applies and ϕ and ψ represent internal spaces of either two electrons or two positrons, and also vanishes when the plus sign applies and we are dealing with mutual antiparticles. Moreover if ϕ and ψ represent free fermions then k is either the sum or the difference of their momenta, so that the above depicted point interactions are unaffected by adding the algebraic sum of the interacting fermions' momenta to the internal photon state.

Now consider virtual fermions connecting to a node in a Feynman diagram. A fermion's propagator contains a factor $\mathbb{1} \pm \frac{1}{m} \gamma[p]$ for an electron (resp. positron) of momentum p , and we must bear in mind that p is now an integration variable spanning the whole \mathbf{P} . Now if p is “off-shell” then $\mathbb{1} \pm \frac{1}{m} \gamma[p]$ is *not* a projection onto \mathbf{W}^\pm . However we recall that the covariant propagator, in which the dependence on time has been eliminated, is actually the sum of two contributions, corresponding to different time ordering of the propagator's nodes. If we keep the time-dependent description, in which particles only “travel forward” in time, then a fermion's propagator contains a factor

$$\mathbb{1} \pm \frac{1}{m} \gamma[\mathbf{E}_m(p_\perp) + p_\perp] \equiv \mathbb{1} \pm \frac{1}{m} \gamma[p_0 + p_\perp],$$

where the plus is for electrons and the minus is for positrons. This is the projection onto \mathbf{W}_p^\pm with $p \equiv \mathbf{E}_m(p_\perp) + p_\perp \in \mathbf{P}_m$, hence the above arguments can be extended to this situation.

Finally, we note that these results can be straightforwardly extended to more general gauge theories of the type described in §1.2.

3.4 Concluding remarks

In quantum theory all fields, including gauge fields, must be sections of some vector bundle. This requirement can be understood at least from two different points of view. In the “momentum representation”, as sketched in §2.2, the construction of the distributional bundle describing quantum states requires a vector bundle over particle momenta, whose fibers describe the particle's “internal states”. In the “position representation” the fields are (generalized) sections of a vector bundle $\mathcal{O} \otimes \mathbf{E} \rightarrow \mathbf{M}$, where $\mathbf{E} \rightarrow \mathbf{M}$ is the classical configuration bundle (a vector bundle whose sections are the fields of the theory under consideration) and \mathcal{O} is an infinite-dimensional \mathbb{Z}_2 -graded algebra, generated by absorption and emission operators.¹⁰

¹⁰The relation between these two approaches in terms of F-smooth geometry will be examined in a forthcoming paper.

According to a third, quite different point a view [22, 12], *the system defines the geometry* and reality is fundamentally discrete; any notion related to continuity should be recovered as a convenience in the description of sufficiently complex systems. Ideas of this kind have been around for some time and have inspired a few tries at serious theories [23, 26, 25], but, as far as I know, no definitely convincing results have been obtained. In Loop Quantum Gravity, in particular, certain discrete geometric structures are the basic quantum states, but how ordinary matter enters the scheme is still unclear. By contrast I propose that physical reality *is* fundamentally a network, whose nodes and edges we call *events* and *particles*, respectively. In a sufficiently large portion of the network, approximate geometric relations will emerge among its external edges; on the other hand, knowing about some external edges we can guess at other external edges in probabilistic terms. So spacetime, gravity (*not* quantum gravity) and quantum mechanics could all emerge from a more fundamental discrete theory.

The relation between spin and spacetime geometry supports these ideas. Rather than trying to recover Euclidean geometry from general spin networks, we could focus our attention on networks whose edges are labeled by internal states, taking the relations among spin and particle's momentum (§3.2) into account. A possible way of undertaking this task is to try and immerse these networks into a manifold with suitable properties, to be chosen in such a way to allow us to derive experimentally testable consequences. Since a measure apparatus is macroscopic, such consequences must be of statistical nature.

Spacetime metric and bundle connections belong to the macroscopic notions which allow us to handle the physics; in this sense they could be viewed as “mean field” background properties of a physical system. Gauge particles, in particular, are related to connections, as the relation must consider the partial indeterminacy of the particles' internal states when expressed in terms of spacetime geometry.

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